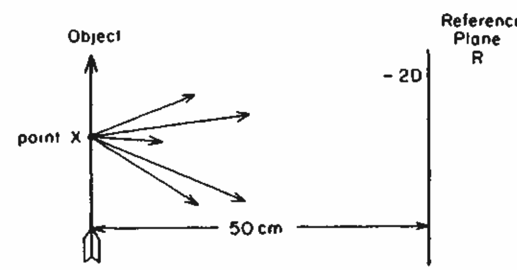
VERGENCE AND LENSES

Vergence

Any object — a colorful vase, a black-and-white photograph, a stick, or a letter E on a Snellen chart — can be considered to be made up of an infinite number of points, each of which contributes something to the overall make-up of that object. To enable us to form an image of that object, we must have some light coming forth from it. The amount doesn't have to be very great; in fact it can be very little, but there has to be some. That light energy can be emitted by it, reflected from it, transmitted through it — but somehow light energy (which may even be invisible to the eye) must be given off. The object is then considered "luminous." From it, a myriad of light rays are thrown off; each is infinitely thin and each projects to a specific *direction*.

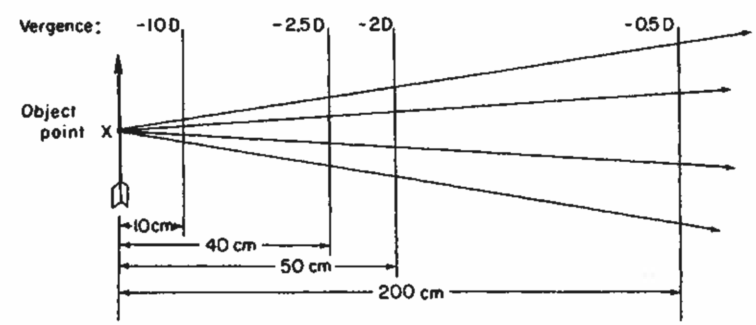
Before we can deal with these rays intelligently, we all must agree on certain of their characteristics. These have to do with light *conventions* and are simply our rules for dealing with these "animals." One convention is that the light must travel from left to right. Of course, light travels in *all* directions, but for our analyses here, please just accept this first convention about the light ray.

Light rays will always diverge (spread apart) from *every* point comprising a real object, so let's focus our attention on the rays emitted by *one* point — any point X — on that object.



From point X in the above figure five rays out of the many possible ones are drawn and travel to the right. These rays will arrive at a reference plane, say, 50 cm away. When they strike this plane, they are diverging at a certain rate. We define (not explain) that, *at* the reference plane, the *vergence* of those rays from point X is inversely proportional to the distance (in meters) between X and the plane. That is, at reference plane R, the rays from point X have a vergence of $\frac{1}{.5 \text{ meters}} = 2$ Diopters. Again by *our* convention, all rays *diverging* at any reference plane are considered negative (—) so the actual vergence here is written as — 2 Diopters.

All rays from *all* real points *di*verge. When you run across rays which are *con*verging, please realize that they must have been created by some other optical system; they cannot occur naturally. These are defined as positive (+) by convention. We will adhere to this sign convention throughout this book. But, please do not consult other texts to confirm this. You may find that they do not agree and you will be confused. Just learn it this way. Again, minus means *di*verging rays and plus *con*verging, whether talking about lenses, mirrors, eyes, or any optical system at all. Simple enough.



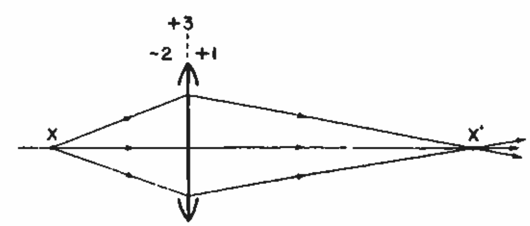
If we move our reference plane closer to point X so that it is only 40 cm away, the vergence at that plane is now — 2.5 Diopters. $\frac{1}{.40 \text{ meters}}$; at 10 cm, it is — 10 D, and at 2 meters, it is — 0.5 D. Do not forget that minus sign.

If we remove the actual plane entirely, we can still talk about the vergence of the rays *at a given distance from X;* this vergence will be the same whether or not there is a real plane there to intercept the rays.

The Plus Lens

Let us now put a lens — any lens — somewhere to the right of our luminous point X — say, at 66 cm. The vergence of the light from X as measured at the lens plane is — 1.5 D (the same as its vergence at that position if the lens were not there). The lens has certain properties which bestow upon it the ability to *change* the vergence of those light rays falling on it. Its ability to change vergence is also expressed in diopters. A lens is considered plus (+) if it *adds* vergence to the incoming light, and minus (—) if it *subtracts* vergence (or makes the light more *di*vergent.)

If we place a + 3 Diopter lens 50 cm to the right of point X, the lens will add + 3 D of vergence to the incoming rays which already have a vergence of — 2 D at that lens plane. The light rays will leave the lens with a vergence (again, as measured *at the lens plane*) of + 1 Diopter. See the diagram below:



To keep us aware of the fact that our vergence measurements take place at the lens plane, we write a — 2 on our diagram just to the left of the lens, as the incoming rays hit it. This is the vergence of object point X and will be labeled *U* (in diopters). The + 3 Diopters of lens power *P* combines with (is added to) this object vergence, and the resulting *image-point* vergence, V, of + 1 Diopter is written just to the right of the lens (again to remind us that the *lens* is the reference plane). What we have just said can be written in math shorthand:

$$U + P = V$$

The object vergence combined with the vergence-change provided by a lens yields the image vergence. This is one of the very few key "formulas" you *must* know; we will continue to rely on it heavily.

Again by convention, *vergences* will be expressed in capital letters: U for the object, V for the image, and small letters u and v corresponding, respectively, to object and image distances:

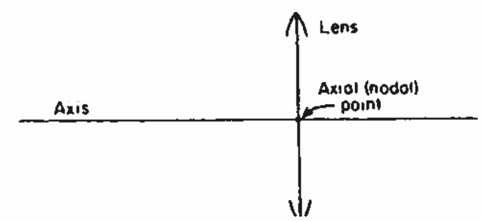
$$\frac{1}{U} = u,$$

$$\frac{1}{V} = v$$

The previous object-image example is our first encounter (here, in the *image*), with converging or + rays. We see that it took an optical system to create them. The image vergence of "+ 1 D" says that the rays which leave the lens P are *con*verging (the + sign) and do so towards an image point located $\frac{1}{V}$ meters to the right of lens P. Since we know the image vergence in diopters is + 1 D, the actual distance to the image point is $\frac{1}{+1} = +1$ meter. Thus, the point X' (the *image* point which corresponds optically to the real object point X) is located 1 meter to the right of lens P. The image point is a *real* one, that is, it can be focused on a screen.

X and X' are called *conjugate* points — they are related by their being the object and image of one another. Since light rays are completely retraceable, if the real object point were located instead exactly at X', its image by lens P would always be precisely at X. However, we have agreed (by our convention) to stay away from light rays which move from right to left, but please realize they *can* do so. (A mirror *would* reverse the direction of these rays, but this will be explored later).

Another definition is now in order. The lens *axis:* this line coincides with and represents a ray of light which falls perpendicular to the lens surface *and* whose direction of travel is not disturbed by the lens; the axial ray goes through undeviated.



The spot on the lens that this ray strikes is the axial point — a very critical one for all our optical constructions. Actually *any* ray from *any* direction that passes through the axial point will also proceed undeviated; we also call this point the *NODAL POINT*. In our "thin" lens system here, the axial point and nodal point coincide exactly. When we study more complex optical systems, we will investigate such a point further.

Recall, we chose X to represent a *single* point in a much larger object composed of many such points; in the optical example given, we could have chosen *any* X that was equally distant from our lens P. It should, therefore, be clear that all such object points must lie in a plane which is perpendicular to the lens axis and parallel to lens P. These points constitute an *object plane*. (*Other* points on the same object but *not* in that same plane would necessarily yield rays of different vergence at the lens.)

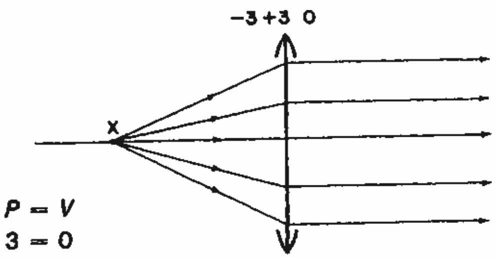
Just as X establishes an object plane, X' (the image point of X) fixes an *image* plane which is also parallel to lens P. All points in the same object plane as X will be imaged somewhere in the same image plane as X'. The exact, corresponding positions of these points will be determined later.

Referring back to our same + 3 D lens P, let's move object X closer toward it. When X moves from 50 cm to 40 cm from the lens, its vergence at the lens increases from — 2 to — 2.5 Diopters.

$$U + P = V \text{ becomes } -2.5 + 3.0 = +0.5$$

The *image* rays are still converging, but the *distance* from the lens to the new image point X' has now increased; X' has moved further to the right, to a position $\frac{1}{.5\text{ D}}$ or 2 meters away.

As X moves even closer — to 33 cm from the lens — something peculiar happens:



$$U + P = V$$
$$-3 + 3 = 0$$

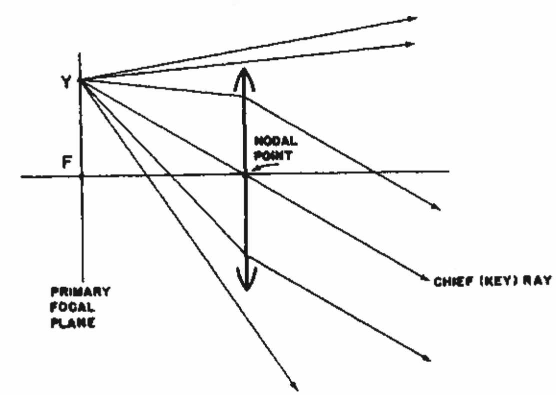
The image point vergence becomes zero; its distance from lens P is v.

$$v = \frac{1}{V}$$

Since it is a mathematical commandment that "Thou shalt not divide by zero" and here $V = 0$, now what? We cannot say that there is no image point to correspond to object point X; but what we *can* say is that as the image vergence V *approaches* zero, v approaches an infinite distance to the right. Thus, image point X′ is located a long distance away — at "right infinity."

Here we have found a special position of object point X relative to lens P — the location of an object point, the image of which is at infinity. That position along the lens axis is called the ***PRIMARY FOCAL POINT*** (F) of the lens. Light rays which originate there and strike the lens will leave the lens in parallel bundles.

That plane which contains F (parallel to the lens plane) is the *primary focal plane.* Any point Y in this plane emits rays which, of course, diverge — those rays that fall upon lens P will enter the "image space" (*following* the lens' action) as parallel to each other; but, parallel to what direction? Certainly not to the lens axis, since only rays arising specifically at axial point F itself would do that. No; that direction taken by these parallel rays in the image will be indicated by only *one* of the rays leaving object point Y — that particular one that leaves Y aimed directly at the axial (nodal) point of lens P (see the diagram below): that ray then (by our definition of the nodal point) will continue on undeviated and will establish the direction of all the parallel rays in the "image space" which arise at point Y in the primary focal plane.



We should now see that the *FOCAL LENGTH f* of the lens (that is, the *axial distance* between F and the lens) is nothing more than that specific object distance *u* which yields an image vergence of zero for this particular lens P. So *f* in meters must equal $\frac{1}{\text{lens power (in Diopters)}}$.

Our next simple "formula" then is

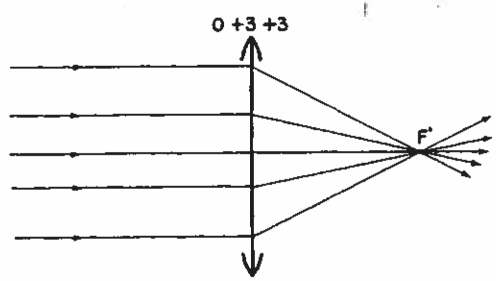
$$f = \frac{1}{P}.$$

A + 14 D lens will have a focal length of $\frac{1}{+14}$ or + 7.1 cm.

If we place a 35 mm slide transparency in the position of the primary focal plane of a projector lens and make the slide very luminous by shining a strong projector light bulb onto it, the lens would project the image of that slide onto a screen. The image would be sharp and clear if the screen were at right infinity. Since that makes for a rather long projection room, the slide can be placed slightly shy of F, that is, a little further from the lens (thus reducing somewhat the vergence of the light rays emanating from each point on the slide); then the projected image would be found at some finite distance in front of the lens — a little more practical than infinity for most projection!

So far, we have been seeing how the image point moves as we brought an axial object point X closer to the lens. Let's backtrack for one moment. When point X on the axis is a long way to the *left* of the lens (at left infinity), the light rays emitted from it are, naturally, divergent. But since the lens is so far away, by the time the rays reach it, they are diverging so minimally that we must consider them arriving in a parallel bundle, that is with an object vergence *U* of zero.

When we look at the lens in the diagram below, we see a number of parallel light rays drawn, but remember, *all of these came from one point*, X, located at infinity on the left and on the lens axis.

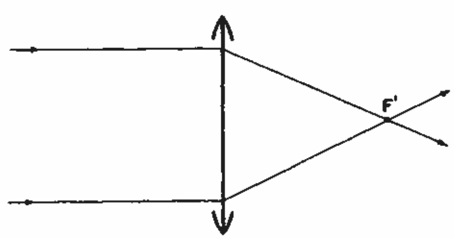


The *image* of axial object point X will be at F' — the *SECONDARY FOCAL POINT* (and another key position). F' is the *image* corresponding (conjugate) to an axial object point situated at infinity.

* * *

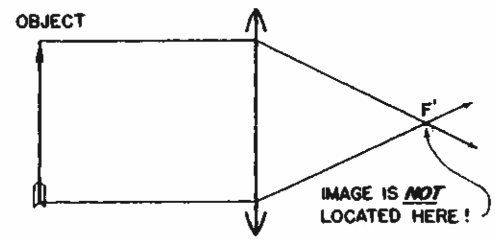
As an interjection, there is one particular type of *erroneous* diagram that is often sketched by beginners, but unhappily, it is also occasionally found in elementary physics or physiology texts when simple optics is being "explained" — I quiver each time I see it, or a reasonable facsimile thereof, and now you should too!

The diagram below is correct;



the two rays drawn, of course, signify light emanating from *one* point (the axial one) of a very distant object as they are brought to a point focus at F'.

Now look at this one:

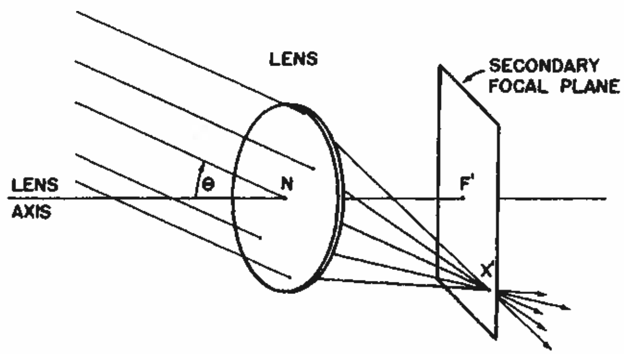


This diagram, with one ray *from each end* of an arrow-shaped object, I believe, actually attempts to show the same thing (that is, a final image at the focal point F'), but this one is horribly incorrect. The true image of the arrow object must be further to the right than F'. Diagrams like this and its ilk have done much to disturb students by shattering their confidence in being able to understand something simple. Don't you make this same mistake. 'Nuf said.

* * *

"Off-axis" Objects and Images

When object point X is at infinity but *above* the lens axis, rays arising there also arrive at the lens in parallel bundles, but strike the lens at some angle (inclination) to the axis. How steep an angle? You guessed it. That angle of inclination θ of all these parallel rays would be given by *the one ray* from point X which was directed at the nodal point of the lens. The image, of course, would be located in the secondary focal plane, since that plane is the home of *all* image points representing object points at infinity. The exact position of the image point in that plane is pinpointed by our undeviated ray through the lens nodal point. (See next figure.)



If a sheet of camera film were located at this secondary focal plane and lens P were a simple camera lens, a sharp image of every object point located at optical infinity would be created on the film and captured for posterity, (if the film were properly exposed).

Let us now continue where we left off and move object point X closer to the lens P. Point X will then be closer to the lens than the primary focal point F. Say, X is at a distance of 25 cm from the lens: then,

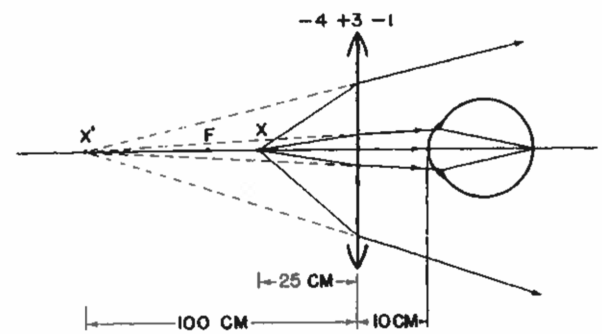
$$U + P = V$$
$$-4 + 3 = -1$$

The image point vergence at lens P is — 1 D.
Here is our first encounter with a negative vergence for an *image* point; how is this to be interpreted?

By convention, we originally agreed that *minus* meant divergent object *or* image rays. So, here too, we must observe this convention, and the minus sign must mean divergent rays are present in the *"image space"*, that is, *after* leaving the lens. Since these rays are diverging, they *cannot* be brought into focus on a screen; only converging image rays can form a *real* image as mentioned before. We say that (by definition) if *diverging* rays are present in the "image space", the image is *virtual* — it cannot be focused on a screen. Even so, there still *is* an image point X' which is conjugate (corresponds) to object point X; but, the image is located on the *left* of the lens! How far? Well,

$$U + P = V$$
$$-4 + 3 = -1$$
$$\frac{1}{V} = v; \frac{1}{-1\text{ D}} = -1 \text{ meter}$$

Thus, the image point X' would be 100 cm to the *left* of the lens.

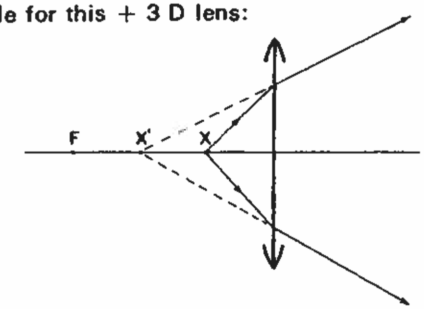


That is, although the *actual* rays (shown in *solid* lines) are divergent as they leave lens P, they simulate rays (dotted lines) which would have come from a real point located 100 cm away. All *real*, honest-to-goodness light rays will, in the diagrams of this book, be indicated as solid lines, while rays from *apparent* (virtual) points will be shown as dotted lines).

For all intents and purposes then, after leaving the lens, the rays will *seem* to arise from X'. Thus an eye looking through lens P would not "see" point X (which is really 25 cm from the lens); it would "see" the image point X' (as created by the lens) as if it were located 100 cm away from it.

If you wish to know the *vergence* at the *eye* of those light rays that seem to come from X', you will have to determine the distance between X' and the eye. Say, the eye is located 10 cm from lens P. Then X' must be located 110 cm from the eye, and X' has a vergence of $\frac{1}{-1.1 \text{ meters}}$ or — .91 D at the eye. Notice now that not only is X' an image created by lens P; it is also, simultaneously, an object for the eye and, thus, can be said to lie in the *"object* space" of the eye. (Even though I have already tossed out the terms "object" and "image" space glibly, I would still like to withhold their definitions until a bit later.)

Consider one last example for this + 3 D lens:



X continues to move closer to P — to a position only 5 cm from it. X presents a vergence U to the lens of $-\frac{1}{.05\text{ m}}$ or -20 D.

$$U + P = V$$

$$-20 + 3 = -17$$

The image X' is now located at $\frac{1}{-17\text{ D}}$ or -5.9 cm, to the left of the lens. Furthermore, the closer X is to the lens, the smaller the apparent separation between X and X'.

Object Movement Vs. Image Movement — Plus

With the help of a table to recapitulate the examples given so far, let us study what happens to the image movement as the object point moves from left-hand infinity towards a + 3 D lens.

Object distance *u* in *cm*	∞	− 50	− 40	− 33 (at F)	− 25	− 5
Image distance *v* in *cm*	+ 33 (at F')	+ 100	+ 200	∞	− 100	− 5.9

As the object located at left infinity moves to the right (towards the lens), the image point first seen at F' also moves towards the right, more and more rapidly until the image reaches right infinity precisely when the object arrives at F. With any further approach of the object towards the lens, the image suddenly appears at *left* infinity — (consider that the image initially moved *away* from the lens to right infinity and then "around the earth" and suddenly appeared from the left.)

The image then proceeds to approach the lens from the left, tagging behind the object as the latter moves still closer to the lens. The image finally catches up with the object *at* the lens plane. Thus with the plus lens, as the object moves to the right, the image moves to the right also.

So far we have examined the properties of a + 3 lens. We can generalize that *any* plus lens will have identical properties, aside from variation in its power. Each lens will exert its own particular influence on the incoming object vergence. A + 12 Diopter aphakic spectacle lens would add + 12 Diopters of convergence to *any* object rays arriving at it. A + 0.001 Diopter astronomical telescope objective would add this tiny amount of *con*vergence to incoming object rays.

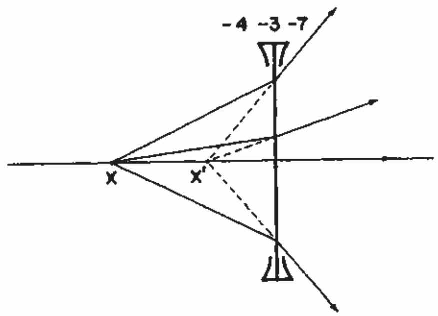
The Minus Lens

A *minus* lens has the ability to add *di*vergence to incoming rays. Our same little relationship $U + P = V$ and all our related sign conventions continue to hold true.

If a real object point X is located 25 cm to the left of lens P of − 3 Diopters, the object vergence at the lens would be $\frac{1}{-.25\text{ meters}}$

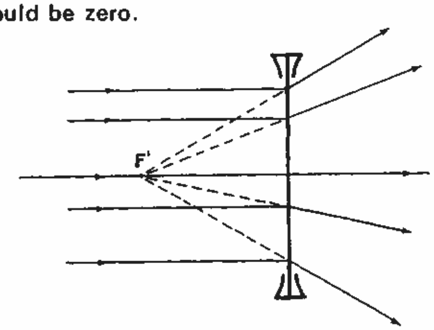
or — 4 D. The — 3 D lens adds *its* vergence power of — 3 D to the incoming — 4 D vergence to yield — 7 D of image vergence.

$$U + P = V$$
$$-4 + (-3) = -7$$



This locates an image point at $-\frac{1}{7\text{ D}}$ or — 14.1 cm, the minus sign again meaning "to the left of lens P".

If the object (in the next figure) were at left infinity, the object vergence *U* would be zero.



$$U + P = V$$

Therefore, $0 + (-3) = -3$

Thus, the image vergence is — 3 Diopters, and the image point would be found at — 33 cm (33 cm to *left* of lens P). We have already defined that the *image* point conjugate to an axial object point located at infinity is called the secondary focal point (F') of a lens so this

minus lens' F' is 33 cm on the *left*.

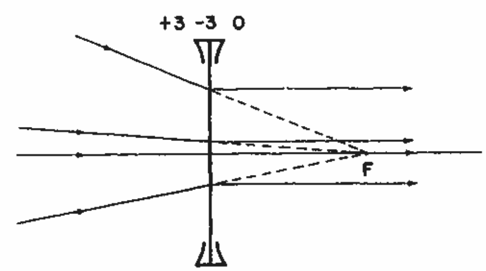
With any minus lens, F' will always be on the left; but remember also that F' is still an *image*, just as much as is F' of a *plus* lens. The object point at infinity, though it appears to be on the same side of the lens as F', it is not in the same "space". The image point F' exists only in the "image" space, the term signifying that we are dealing with light which has already been influenced by a given lens.

Now, how on earth can we locate the *primary* focal point (F) of a minus lens, since by definition it is the *object*, the image of which is located at right infinity (that is, the image vergence must be zero)? Our trusty relationship $U + P = V$ should help us:

$$U - 3 = 0$$

U must $= +3$, which means that this object point F (the primary focal point of the lens) must be 33 cm to the *right* of the lens. Our light rays (remember this convention?) must travel from left to right; so, for us to obtain an *image* vergence of zero, the object vergence has to be plus (convergent) in the *object space*. Only then can parallel rays be created by the minus lens. (As stated early in this treatise, convergent object rays can only be formed by another optical device; they cannot occur naturally.)

To demonstrate F of a minus lens diagrammatically, we must show object rays converging toward F (the "object" point) *before* they impinge on the lens. The fact that the object rays never really reach F but are only directed towards it should not distress you.

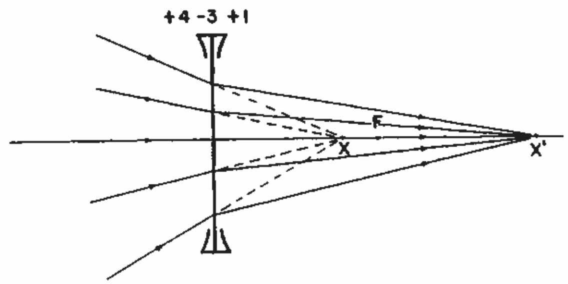


When convergent rays fall on any lens — plus or minus — the object vergence will of course be *plus*, and the object is considered *"virtual"* since it is intangible. Optically, however, a virtual object can be manipulated as readily as a real one, just as can a virtual image. *Virtual* object rays will be shown in the accompanying diagrams as dotted instead of solid lines.

Let us now move the virtual "object" closer to the lens, to somewhere between the lens and F. We will thus have to make the object rays even more convergent than + 3 D. Say the vergence of object point X is + 4 Diopters (see figure below);

$$U + P = V$$

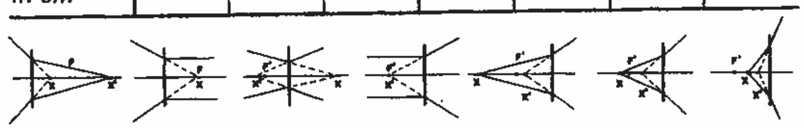
$$+4 + (-3) = +1 \text{ D}$$ and a *real* image point will be created by a minus lens. If you desire, you can focus this image onto a screen one meter away — to the right.



Object Movement Vs. Image Movement — Minus

Again, a table has been constructed relating the object distance *u* and image distance *v* for a minus lens. I have added a few more examples for our — 3 Diopter lens:

Object distance *u* in *cm*	+ 25	+ 33 (at F)	+ 100	∞	— 33	— 25	— 10
Image distance *v* in *cm*	+ 100	∞	— 50	— 33 (at F')	— 16.7	— 14.1	— 6.9



As an object point moves from the right side of the lens towards right infinity (of necessity, these object points are created by *convergent* light beams from some other optical system), the image "moves" ahead of it in the same direction — *through* right infinity, "around the earth" and over to left infinity, and thence toward the lens. With a *real* object point (which moves from *left* infinity towards the lens), the corresponding virtual image point will continue to move ahead of it to the right (still closer to the lens), but also in the same direction as the moving object.

The point of these two tables — one (a few pages ago) for a plus lens, the other for a minus — is to demonstrate that the sign of the lens makes no difference to one clear-cut conclusion: the object and its image *always* will move in the same direction. *When any object moves to the right or left, its image by any type of optical system, plus or minus, will do likewise.* This point is quite important in helping you to understand the movement of images when dealing with the eye and with cases of refractive error.

Object Space Vs. Image Space

Up to now, I wanted you to get the "feel" of handling these terms yourself, since sometimes when dealing with concepts, intuition is better than definition. Now, however, to set the record straight, I will try to clarify the two conceptual terms, "object space" and "image space", both of which I have already used freely. But by now, you already should have an inkling of what they mean.

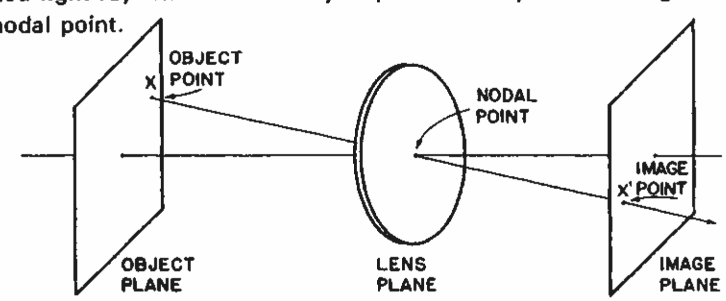
Prior to impinging on any lens, all rays which leave an object plane are said to be located in the "object space." After the lens adds its own vergence to those rays, the resultant vergence is said to exist in the "image space." That sounds easy enough. However, the lens which creates those spaces "looks" like it physically separates the "object space" on the left from the "image space" on the right — it does not. Both spaces co-exist simultaneously on *both* sides of the lens, and that is what makes this concept confusing.

Object space exists only for rays and ray directions *before* they are influenced by a lens. Consider that object space is composed of *all* points which could possibly serve as an object for the lens. We have studied about divergent object rays which come from a real object located on the left of the lens; we have also seen how convergent object rays might optically form a virtual object on the right of any lens. The latter object also is in the object space of the lens. So, we can see that object space does completely surround the lens. Remember then, the object space of a lens may contain real or virtual rays, but *before* they have been influenced by the lens.

Image space, on the other hand, exists only for rays *after* they are influenced by a lens or an optical system; this space exists only because the lens has *already* exerted its influence on some object rays. Image space also completely surrounds the lens; if the *image* is on the right of the lens, it is real, if on the left, it is virtual. In either case, the image is still in the "image space."

You should now understand that every lens is completely engulfed — at the same time — by both object and image spaces. Contrary to appearances, these were not created to confuse you. Keep the above descriptions in mind.

So far, we have looked at *axial* object points and image points and diagrammed their conjugate relationship, but we also elaborated (through our discussion of the axial focal points) that the *points* also represent planes. These object and image planes which contain the corresponding axial points are parallel to the lens plane and possess all the vergence properties (at the lens plane) of those points. One locates these planes relative to the lens the same way as one does the axial points ($U + P = V$). Any object point in one plane has its image in its conjugate plane and, at the specific location denoted by that single, undeviated light ray which leaves object point X and passes through the lens nodal point.

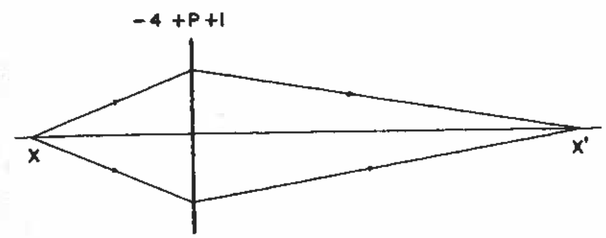


What else can we learn from $U + P = V$?
If we know any two terms, the third can easily be calculated and all sorts of nice little problems become manageable.

EXAMPLE 1

What strength lens will image an object located 25 cm to the left of it onto a screen 1 meter distant to the right?

ANS:



$$U + P = V$$
$$-4 + P = +1$$
$$P = +5\text{ D}$$

A + 5 D lens fills the bill.

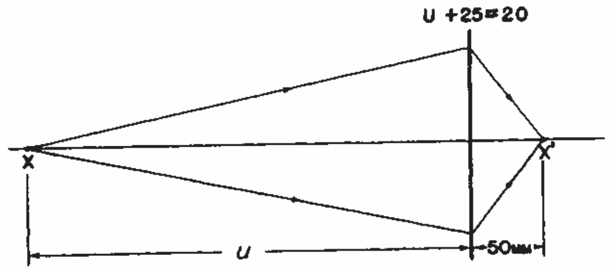
EXAMPLE 2

How far away must an object be to have its image sharply defined on film which is located 50 mm behind a + 25 D camera lens?

ANS: $v = 50\text{ mm} = .05$ meters, and $P = +25$ D.

$$U + P = V$$
$$U + 25 = +\frac{1}{.05} \quad \text{(See next figure.)}$$
$$U = 20 - 25 = -5\text{ D}$$
$$u = \frac{1}{-5} = -0.2\text{ m} = -20\text{ cm}$$

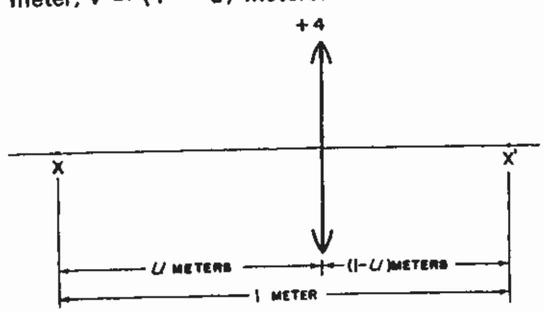
The object X must be 20 cm away (to the left).



EXAMPLE 3

Exactly where, ***between*** an object and its corresponding image, must a + 4 D lens be placed so as to have the object and image separated by 1 meter?

ANS: X and X' are 1 meter apart; u is the object distance and v is the image distance. (See diagram below).
In ***absolute*** distance measurement (without regard to the sign), since $u + v = 1$ meter, $v = (1 - u)$ meters.



We are given that the lens is ***between*** the object and the image, so whatever distance u happens to be, the object vergence U must be minus at the lens plane. With comparable reasoning, V must be plus.

So we have $U = -\frac{1}{u}$ and $V = \frac{1}{1-u}$

Now, substitute these for U, P, and V in our general formula,

$$U + P = V$$

$$-\frac{1}{u} + 4 = \frac{1}{1-u}$$

To solve this equation for u, we must simplify it by eliminating the denominators. This manipulation requires easy 9th grade algebra.

$$-(1-u) + 4(u)(1-u) = u$$

$$-1 + u + 4u - 4u^2 = u$$

$$-1 + 4u - 4u^2 = 0$$

Multiplying by (— 1);

$$4u^2 - 4u + 1 = 0$$

$$(2u - 1)^2 = 0$$

Take the square root;

$$2u - 1 = 0$$

$$2u = 1$$

$$u = \tfrac{1}{2} \text{ meter}$$

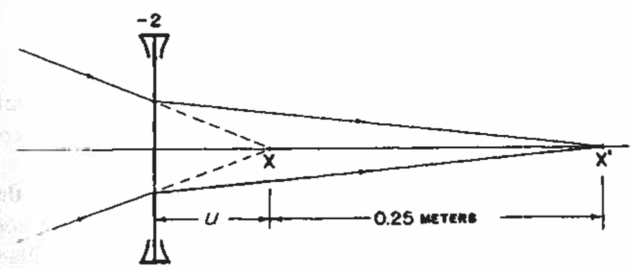
$$1 - u = v = \tfrac{1}{2} \text{ meter}$$

Therefore, the lens would be located 50 cm from both the object and the image, that is, half-way between them.*

EXAMPLE 4

A convergent light beam from a projector located on the left creates a clear, 3 cm diameter spot image on a screen on the right. You don't care about the size of the spot, but you want to move that image 25 cm to the right without budging the projector. You have available a — 2 D lens to help you. Where must you hold this lens to perform this image displacement?

ANS: The distance X to X' is given as 0.25 meters.



Let u = the distance from our lens to the image at X (which would have been produced by the projector if the lens did not intercept the rays first!) This is the object distance for our — 2 D lens. The object vergence $U = +\frac{1}{u}$; $P = -2$; the image vergence $V = +\frac{1}{(.25 + u)}$.

* If the + 4 D lens were not specified as being ***between*** X and X', two other solutions become possible. Can you find them?

$$U + P = V$$

$$+\frac{1}{u} - 2 = +\frac{1}{.25 + u}$$

$$(.25 + u) - 2(u)(.25 + u) = u$$

$$.25 - .5u - 2u^2 = 0$$

Multiply by (— 1):

$$2u^2 + .5u - .25 = 0$$

Factor:

$$(2u - .5)(u + .5) = 0$$

Thus we have *two* answers to the problem:

a) $2u = .5$

$u = .25$ meter

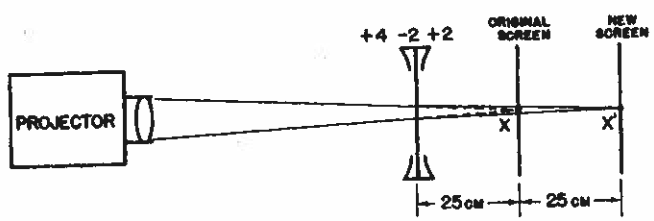
$u = +25$ cm

b) $u = -.5$

$u = -50$ cm

How do we interpret these answers?

a) The — 2 lens should be held 25 cm in front of the original screen. This would move the projected image to a position 25 cm further away.

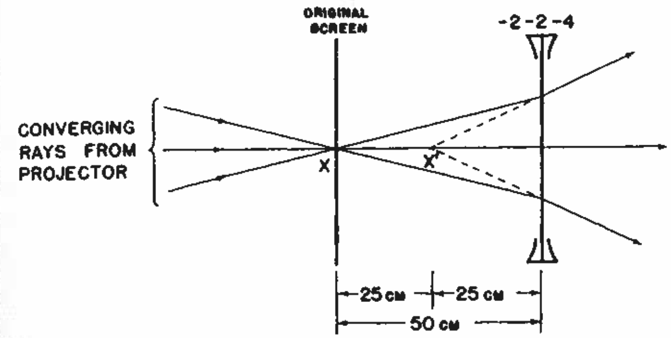


b) But, we see from our algebraic solution that another possibility is suggested (see figure below). An object distance u of — 50 cm tells us to hold the lens in such a way as to present a diverging beam to the lens. This is possible only if we place it in the light path *after* the projection image is formed at X. When there is no screen in this position, the rays will continue on, but then they will be *divergent*. Placed at — 50 cm, the — 2 D lens "sees" an object vergence of — 2 D and creates an image vergence of — 4 D, since

$$U + P = V$$

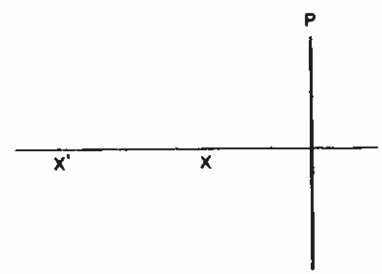
$$-2 + (-2) = -4$$

An image vergence of — 4 signifies that our lens creates a virtual image located 25 cm to its *left.*

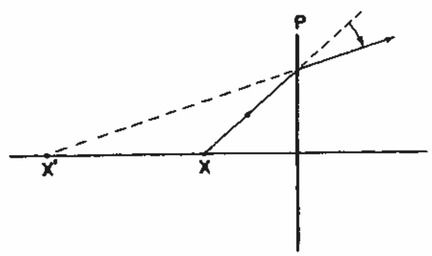


The original object point X would indeed be imaged at X', 25 cm further to its right, as was the condition stated in the original problem, but of course, we had no way of knowing *a priori* that the math solution would include a *virtual* final image. Though this answer fulfills the image criteria, it is impossible to focus this image without additional converging lenses, so this answer is not a practical solution, while answer (a) is correct *and* practical.

OTHER EXERCISES —

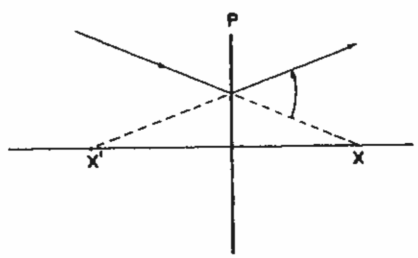


If I asked you to tell me what type of lens (plus or minus) would image axial point X at X' in the above situation, you *should* be able to respond. If you can't, follow along with my reasoning: Since X is on the left of lens P, light rays will diverge toward the lens. After being influenced (refracted) by the lens, all rays from X must be bent as if they came from X'. So, draw *any* ray from X and notice how the lens bends it so that it would appear to come from X'. If it converges the ray, the lens must be plus; If it diverges it, the lens must be minus.



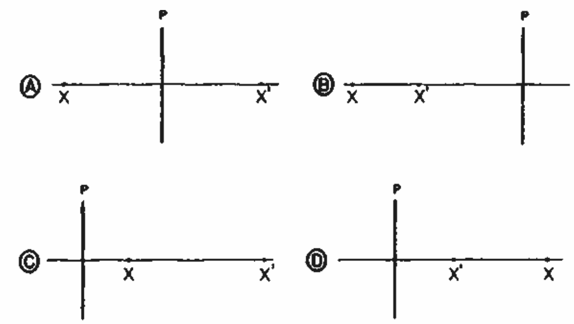
P here must be plus since the light ray is bent downward, towards the lens axis — that is, convergence is added by the lens. The added convergence though was not sufficient to form a *real* image point. X' is thus virtual.

What if X is on the right of lens P?



The only difference with X on the right is that this signifies that light is *convergent* (at the lens plane) toward object point X (which must thus be virtual). After a ray drawn toward X hits lens P, it will act as if it were directed from axial point X'. Again note only what happens *at* the lens to that single light ray. Since the light ray is bent upwards, away from the axis, divergence must have been added by it; therefore P here must be a *minus* lens. In this example both X and X' are virtual!

Try to label a few other lenses as plus or minus:



ANSWERS: A. Plus B. Minus C. Minus D. Plus

MULTIPLE THIN LENS SYSTEMS

When you have to find object-image relationships through more than one lens, you must treat the vergences separately for each lens in succession, always dealing first with the first lens to encounter the incident light. The image position created by the first lens will then be the object position for the second.

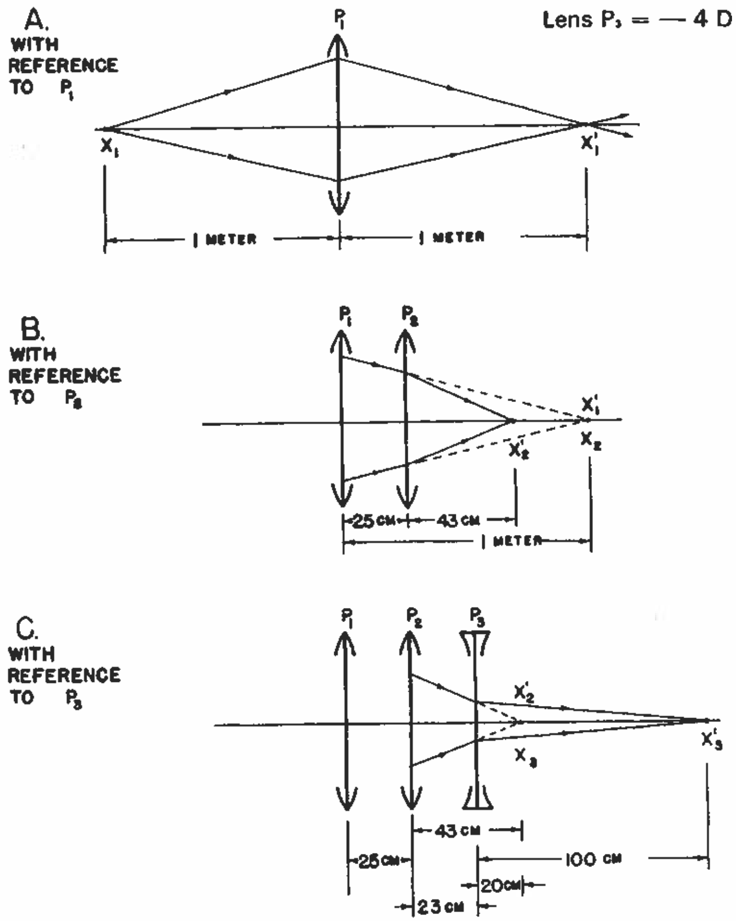
But be careful here; we said we are dealing with *positions*. The image *vergence* created by P_1 (the first lens in a series) is *not* the same as the object vergence for P_2 unless the lenses are in contact; you must *calculate* the appropriate vergences from the known positions. So first, locate the *position* of the image created by P_1; then find the distance of this image (now the object) from P_2. The reciprocal of this distance ($\frac{1}{\text{distance}}$) is the object vergence presented to P_2, and don't forget the *sign* of this object vergence.

EXAMPLE: (Consult the figure where each lens is considered independently):

Lens $P_1 = +2$ D
Lens $P_2 = +1$ D
Lens $P_3 = -4$ D

A real object is 1 meter to left of P_1.
The distance between P_1 and $P_2 = 25$ cm.
The distance between P_2 and $P_3 = 23$ cm.

Lens $P_1 = +2$ D
Lens $P_2 = +1$ D
Lens $P_3 = -4$ D



A. *With reference to Lens P_1:*

$$U + P = V$$
$$-1 + 2 = +1$$

Therefore, the image is located 1 meter to right of P_1 (at X_1').

B. But P_2 is situated 25 cm from P_1; therefore, the image rays from P_1 will strike P_2 before the real image is formed further to the right. As far as P_2 is concerned, it "sees" incident (object) rays aiming at X_1', which is still 75 cm further on to the right. Hence, the incident vergence of those rays at P_2 is $+\frac{1}{0.75 \text{ meters}} = +1.33$ D.

In summary, the image (X_1') formed by lens P_1 with a vergence of $+1$ D now becomes an object (X_2) for lens P_2 *but* with an object vergence at P_2 of $+1.33$ D.

With reference to Lens P_2:

$$U_2 + P_2 = V_2$$
$$+1.33 + 1 = V_2 = +2.33 \text{ D}$$

So the image vergence after P_2 is $+2.33$ D.

$$\frac{1}{V_2} = v_2$$
$$\frac{1}{V_2} = \frac{1}{+2.33}$$; therefore, $v_2 = +43$ cm, that is, 43 cm to right of P_2.

C. Since P_3 is located only 23 cm from P_2, the rays will strike P_3 before the real image is formed. The object distance to P_3 is equal to (43 cm — 23 cm) or 20 cm to the right of P_3 and thus, the object vergence (U_3) presented to P_3 is $+5$ D.

With reference to Lens P_3:

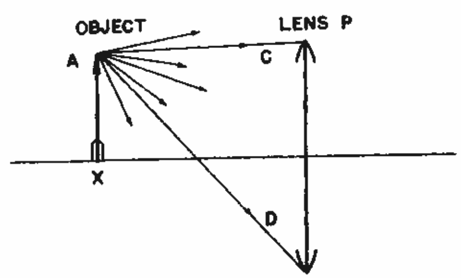
$$U_3 + P_3 = V_3$$
$$+5 - 4 = +1 \text{ D}$$

The final image vergence is $+1$ D; thus, the location of X_3' — the final image formed by the combination of all three lenses — is a *real* one, 1 meter to the right of P_3.

So much for locating by simple algebra the locations of conjugate objects and images relative to single or multiple thin lens systems. One should also be able to determine readily these same relationships by an accurate, *graphical* method. This will be taken up now.

GRAPHICAL ANALYSIS

We have clearly spelled out that an infinite number of light rays will emerge from each point making up any luminous object. Let us concern ourselves with the rays which leave Point A, the tip of an object which sits on the axis of lens P.



In the diagram, light rays are shown being emitted by point A. Many of these rays will fall on lens P — those emitted between the extreme, "limiting" rays C and D which just barely hit the edge of the lens. Each ray will be bent a different amount, but still towards the same image point. If we knew the exact path taken by any of the rays after refraction, we could follow those and find where any two of them intersected; we would then have localized *the* image point corresponding to object point A. Luckily, we *do* know the precise path of three particular rays; (we have already dealt with these). We know the exact road these rays travel and can utilize them in a simple graphical analysis. These three "known" rays in any optical system are as follows (see next diagram):

1) The *"chief"* ray; that particular ray from A that aims directly for the lens nodal point will continue undeviated.

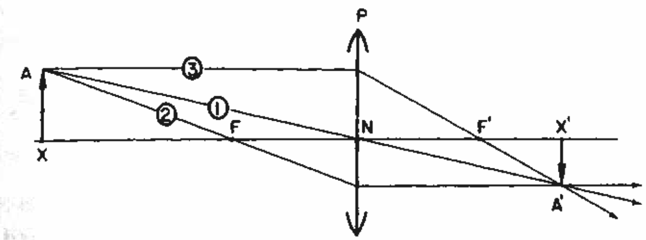
2) The ray through the primary focal point F; we know that *all* rays *originating* at F must leave the lens parallel to the axis. Thus *any* single ray *from any object* which happens to pass through F will also leave lens P parallel to the axis.

3) The ray parallel to the lens axis; *all* rays which are parallel to the lens axis in the object space must, in the image space, pass through the

secondary focal point F' of the lens. Thus, if we select *the one ray* which leaves object point A directed parallel to the lens axis, it also must, after refraction by lens P, be bent toward F'.

Given lens P with

N = nodal point
F = primary focal point
F' = secondary focal point



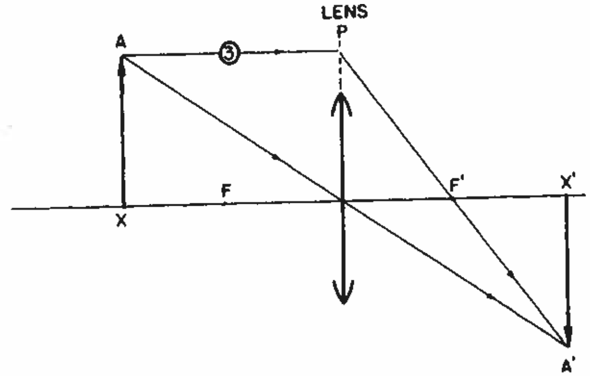
Ray 1 drawn through N is undeviated.
Ray 2 drawn through F must emerge from P parallel to axis.

These two rays cross at A' and establish it as the image point of object point A. To check, continue with ray 3.

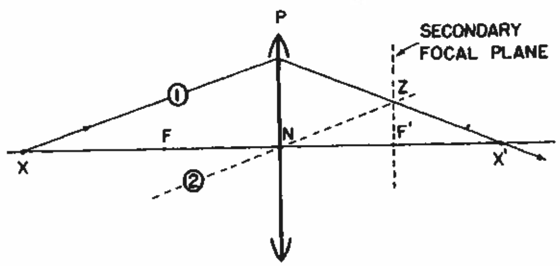
Ray 3 leaves A parallel to axis in the object space and must pass through F' in image space. We see that all three rays intersect at A', and firmly fix A' as that image point conjugate to A. Object point X on the axis is located at the intersection of a perpendicular dropped from point A. X' is similarly located in the image space once A' is determined.

With *any* given lens and a given object point, we should easily be able to construct graphically the corresponding image point (and vice versa!) Remember, any two of the above mentioned three rays will suffice. For practice here, you should draw all three.

Another aid: If you find that in drawing rays 2 or 3 that the lens diagram is not large enough in diameter, simply extend the size of the lens with a dotted line (as in the figure below). All refraction of light rays can schematically be considered to take place somewhere in the lens plane, even if there is no actual lens material there! For example, to draw ray 3 parallel to the lens axis, you must extend lens P diagrammatically upward as shown here.



When we are given only an *axial* point X and wish to find the corresponding image point X', the three ray technique described above will not "work"; all three of the particular rays we have enumerated are still there, but happen to be superimposed upon the line representing the lens axis, and so, we would be unable to locate any *intersection* of rays to find point X'. A slightly different technique can help us, however.



From X draw *any* arbitrary ray 1 which intersects lens P. We know that any and all rays in the object space (even though they do *not* arise at X), if parallel to ray 1, will have to come to a sharp focus somewhere in the secondary focal plane. Let us draw such a ray, 2, parallel to 1, but through N, the lens nodal point. Ray 2 must then be undeviated. (Remember, we can draw this helpful ray even though we know it does *not* denote a *real* ray originating at X. It is simply a logical

device to let us fix point Z in the secondary focal plane.)

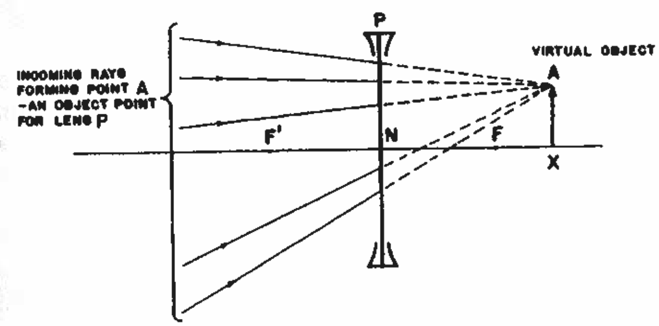
Since all rays parallel to 1 and 2 in the object space must pass through Z in the image space, ray 1 also must be bent by the lens to pass through Z; thus, ray 1 is given a definite *direction* in the image space. But, we need the intersection of *two* rays to determine image point X'. Can you see where we can find the second ray? You should. That second ray is the one which travels along the lens axis itself; we know both object X and X' must lie along this ray. Thus, the intersection of ray 1 and the lens axis establishes X' as the image point of X.

Let us use our same 3 basic rays to indicate graphically the image created by a minus lens with a *virtual* object, a somewhat more complicated, but still very useful exercise.

Given AX — a virtual object on the right lens of P, a *minus* lens. Locate the image graphically:

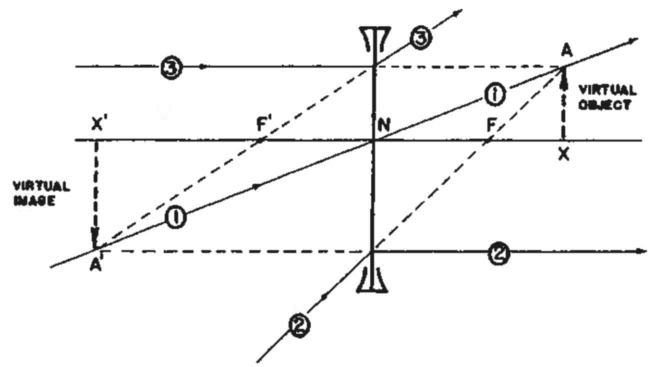
Solution (see figures below):

To establish object AX on the right of lens P, the rays must be converging as shown.



(See next figure.)

1. Of all the rays aiming for point A in the *object* space, select the one ray, 1, which is directed towards A and passes through N. That ray must be undeviated.



2. Select ray 2 (another of the rays aiming for A) which passes through F, in the "object space" even though it appears on the *right* of the lens. This ray must, after refraction, emerge from lens P as a ray parallel to the lens axis. (Watch this next move since it is a bit tricky!) If the direction of ray 2 *after* refraction were extended backward (now within the image space), it would intersect ray 1 (also in the image space) at A', which must therefore be the *image* of A. To check this point, utilize ray 3.

3. Ray 3 is selected as that ray aiming for A (in the object space) which is parallel to the lens axis. After lens refraction, this same ray must pass (or seem to come to) F'. Extended even further back from F', ray 3 also will intersect rays 1 and 2 precisely at A', and confirms its location.

Rays 1, 2 and 3 after refraction (in the image space) seem to come from A' (a virtual image) and any eye (or optical system) peering back through lens P to receive these rays will see A'X' exactly located as diagrammed here.

If one fully grasps the principles underlying the use of the three rays to construct a display of the object-image relationship, he need have no fear in facing *any* construction which might be encountered later.

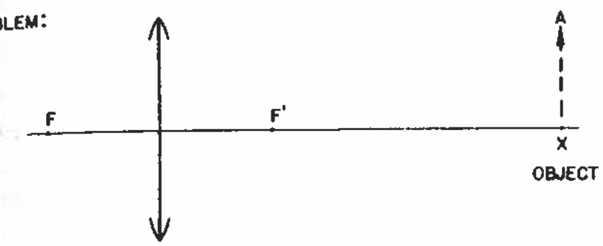
Try two more construction problems: If you find you are unable to draw these out for yourself or you cannot understand the answers

given, please go back and reread this section on graphical analysis.

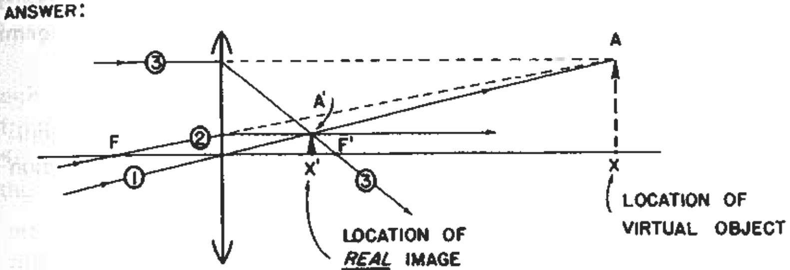
GIVEN: The lens position and the object position and size.

FIND: The position and size of the image in both problems.

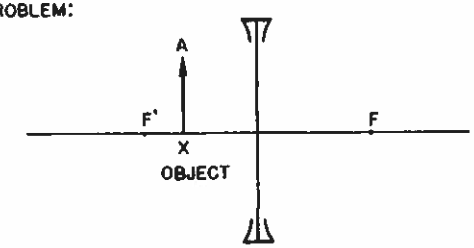
PROBLEM:



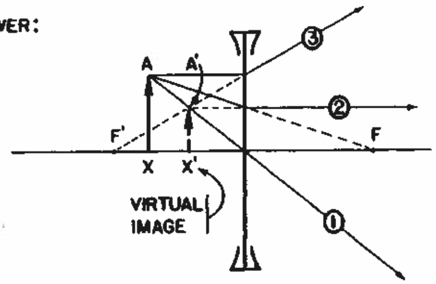
ANSWER:



PROBLEM:



ANSWER:



If you wish to construct the rays from an object through *multiple* thin lenses, you must use the same method shown here for *each* lens separately and consecutively. First locate the first image by lens 1. That image then becomes the object for the second lens; this lens forms another image which is "seen" by the next lens and becomes its object, etc. The final image is fixed when all lenses have exerted their own particular influences on the original object rays.

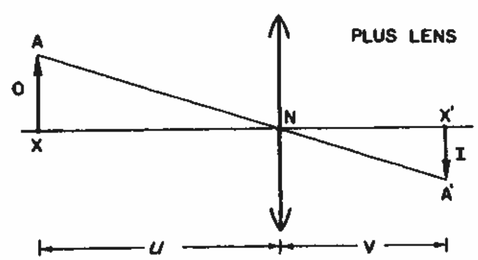
LINEAR MAGNIFICATION

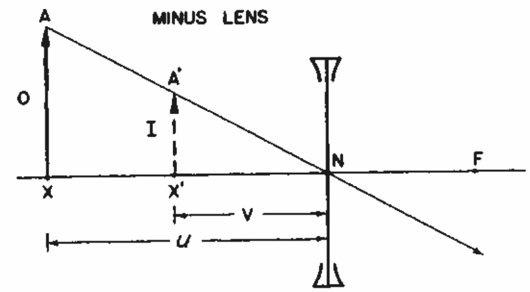
Though we will get into a more complete discussion later of magnification as it pertains to vision, we should mention *linear* magnification here to complete our image construction section.

So far we have not mentioned anything about the relationship of the linear size of the object to that of the image. That is, when object point A is off the lens axis, how far off the axis is the image? Simple geometry tells us this answer in all cases.

Aside from the lens axis, the only other ray necessary for us to see this relationship clearly is ray 1 — that through the nodal point.

These rays establish 2 similar triangles AXN and A'X'N.





No matter what the power of lens (plus or minus) — no matter where the objects and images are in relation to the focal points — these triangles will always be similar, even, of course, if the object and image are on the same side of the lens (as in the figure above).

You should, therefore, be able to see that the *sizes* of the object and image are always directly proportional to their *distances* from the lens. The LINEAR MAGNIFICATION (M) is hereby defined as the ratio of the size of the image to the size of the object. Refer now to the above diagrams:

Image size I = A'X'
Object size O = AX
image axial distance v = NX'
object axial distance u = XN

By definition,

$$M = \frac{I}{O} = \frac{A'X'}{AX}$$

But since the triangles involved are similar,

$$\frac{A'X'}{AX} = \frac{NX'}{XN}$$

Thus; $$\frac{I}{O} = \frac{v}{u}$$

Since $$v = \frac{1}{V} \text{ and } u = \frac{1}{U},$$

$$\frac{v}{u} = \frac{U}{V}$$

So, we have three different, but exactly equivalent, ways of expressing the linear magnification M;

$$M = \frac{I}{O} = \frac{v}{u} = \frac{U}{V}$$

M may be greater, equal to, or less than 1, depending on whether the image size is (respectively) larger than, equal to, or smaller than the object size.

That's all there is to it; and it makes no difference whether the objects or images are real or virtual. Continue to use our same sign convention for vergences and distances and you will find that when Magnification Power turns out to be *minus*, this will always indicate that the image is *inverted* compared to the object, while plus "says" the image is upright.

To find the magnification M, you can use either a geometrical construction (to a set scale) or our simple $U + P = V$ relationship. The algebraic expression is obviously the easiest and most direct method to use routinely.

PROBLEM: What is the overall magnification and the actual size of the projected image produced by a 5 cm focal length projection lens using a 35 mm width slide transparency located 6 cm from the lens. The image is sharply projected on a screen.

ANSWER:

To obtain the Magnification, we must know the image distance (or its vergence) at P.

$$U + P = V$$

$$-\frac{1}{.06} + \frac{1}{.05} = V = \frac{1}{v}$$

$$-16.7\text{ D} + 20\text{ D} = V$$

$$+3.3\text{ D} = V$$

$$\frac{1}{V} = v = +30\text{ cm.}$$

So, the screen is 30 cm to the right of the lens; $u = -6$ cm
$v = 30$ cm

a) $M = \frac{v}{u} = \frac{30}{-6} = -5\text{ X}$

Or even simpler, just use the vergences U and V themselves:

$$M = \frac{U}{V} = \frac{-16.7\text{ D}}{+3.3\text{ D}} = -5\text{ X}$$

This means the image is 5 times the size of the object; the minus sign signifies that the image is inverted related to the object.

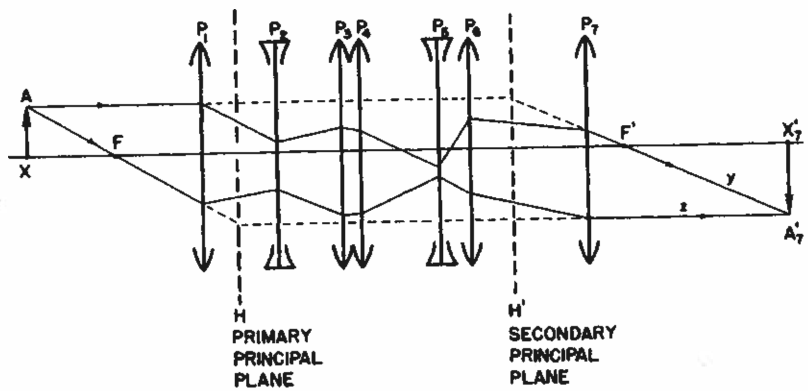
b) Since the object is 35 mm wide and the magnification is 5 X, the image is 5 *times* 35 = 175 mm wide.

PRINCIPAL PLANES AND POINTS

I would like now to introduce another set of terms to complete (*not* complicate) our introduction to the nomenclature applied to optical systems. I will expend slightly more space on this subject than it warrants for the level of optics required by students, but so many budding ophthalmologists continue to ask me to explain (not just define) the concept of *principal planes* that I felt it would save time in the long run to do so here.

When we were studying image formation by a series of thin lenses, we showed that we could locate the final image by either numerical or graphical means, but only after processing object rays successively through *each* lens element in the total system. We will now demonstrate diagrammatically a way to simplify the situation, even with a complicated optical set up. (See the next figure). We will then eliminate all the refracting elements and replace them with two theoretical (though mathematically proper) "refracting" planes. The positions of these planes will be determined. These planes will permit us to neglect all the lenses shown in the figure; that is, each ray emanating from object point A will be able to be treated as if it were influenced only by these two planes. These key reference planes are called the PRINCIPAL PLANES — one primary and one secondary. Their intersections with the lens axis are correspondingly called the PRINCIPAL POINTS.

To see how these planes are located, let us begin with a diagram; but don't let it frighten you. I have *purposefully* chosen a rather complex system of seven assorted lenses to demonstrate how these reference planes can simplify the optical considerations.



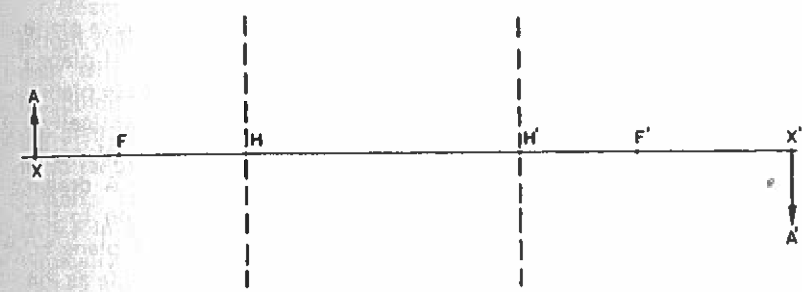
Somehow, by any mechanism we wish (optical bench experiments or actual calculation), we locate F, the primary focal point of the entire system (not just of lens P_1), and F', the system's secondary focal point. We succeed in imaging object AX through all seven lenses to its final position $A_7'X_7'$; so, A_7' is fixed. Now, let us draw the one ray from A to lens P_1 which passes through F. This ray will then go through the entire optical system. (To know the *exact* path of this or any ray through any such complex system, one would have to go through a process of "ray tracing", determining the influence of each lens on the ray. So, let us not worry about the exact path through the system.) Suffice it to say, after assorted bumps and grinds, the ray leaves the last lens P_7 headed directly for A_7'. This final ray will be parallel to the axis since it originally went through F, the system's primary focal point. The final ray is shown as ray Z.

For our analysis it will make no difference what actual bending and flexing this ray has been subjected to in its passage through this complex system. If we simply extend ray Z straight back *as if it were uninfluenced by any lens,* it will somewhere intersect the extension of our original object ray which passed through F of the lens system. This intersection point determines a plane H (dotted in the diagram) which we make perpendicular to the lens axis. It should be obvious that our light ray from A (through F) could be considered as being refracted "by some mystical power" located in plane H, whence it would

leave parallel to the axis and later arrive at A_7'. Any plane so constructed is named the *primary* principal plane.

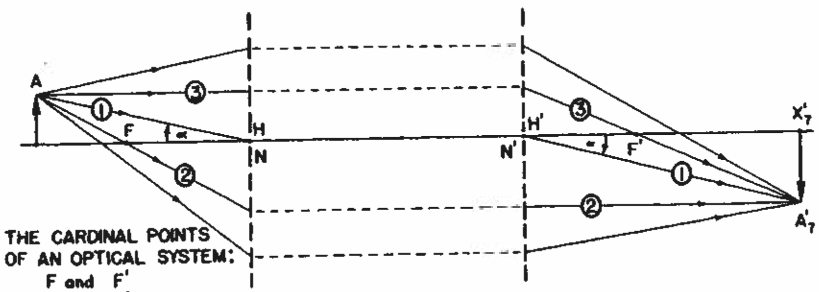
Back to object point A: Also emitted from A is one ray which leaves parallel to the axis. It also is refracted by the entire lens system and finally emerges from lens P_7 aiming towards A_7'. This final ray must cross the axis at F'. This latter ray (Y in the diagram) is extended further backward to intersect the extension of our original object ray (the one that left A parallel to the axis); this intersection determines another plane — the *secondary* principal plane — which is a "surface" acting as if the final refraction took place there. This plane is labeled H' in the diagram.

These two planes, then, can be considered and treated as if they replaced all the other optical elements. This is schematically shown in the figure below:



The positions of these planes can be accurately determined by mathematical computations, but these become more arduous as the lens systems become more complex; they *are* relatively easy to establish on an optical bench in the laboratory, if one wishes.

You can see that there is a physical separation apparent between planes H and H'. Here the space seems large; other times the gap is very small. In either case, in geometrical constructions this gap is treated *as if it weren't there,* and any ray from object A which hits plane H, leaves plane H' at precisely the *same height* — as if the planes were in contact. (You will not usually see any rays drawn within the space HH').



THE CARDINAL POINTS OF AN OPTICAL SYSTEM:
F and F'
H and H'
N and N'

Since any incident ray impinging on plane H will always leave plane H' at the same height from the axis, we say that the principal planes exhibit the property of *unitary linear magnification.* (Also, these planes are actually *conjugate*, that is, one is the image of the other, optically.)

In the above figure, ray 2 from A through F will arrive at plane H; it will leave from plane H' parallel to the axis. Ray 3 from A drawn parallel to the axis will leave plane H' directed to F' and on to the image A_7'. Ray 1 from A, which aims for the axial point of plane H, exits from plane H' also at the axis *and* at exactly the same angle as the incident ray! This should call to mind the "chief" ray which, when we dealt with the simple thin lens, passed undeviated through the lens *nodal* point. (The close relationship between nodal points and principal points will be discussed shortly. They are *not* identical, though here, where we have labeled H and H' as the *axial* principal points of their corresponding planes, these same points also represent the nodal points of *this* system; that is, N and N' happen to be superimposed on H and H'.)

In a complex optical system such as this (actually, in *any* optical system), the principal planes are *the* reference planes; all object and image distances are measured relative to them. So also, the primary

and secondary focal lengths are measured to H and H' and are defined as FH and H'F', respectively. But this requires that the position of H and H' be known; if so, distances FH and H'F' will provide a measure of what is called the *true* or *equivalent* focal length — the reciprocal being the *equivalent power.* Though this latter unit is generally conceded to be the *standard* description of a lens' power, it is *not* a clinically useful unit, because the positions of H and H' are not readily found — they are "intangible".

So, when the principal planes are not identified or localized for you, the focal lengths must be measured in reference to some other, more *convenient* surface, like the axial point on the anterior surface of the lens; its distance from F would then be called the "anterior" focal length. Better yet, measure from the axial point on the *posterior* surface of the lens to F', the secondary focal point. *This* distance is called the "posterior" or "back" focal length.

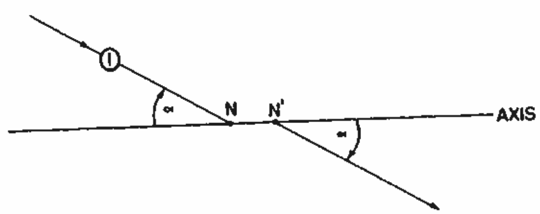
Become familiar with the "back" focal length; *it* is the lens focal length that is implied (when not stated to the contrary) whenever you deal with *ophthalmic* lenses. The clinical instrument known as a lensometer measures this "back" focal length of an "unknown" lens; the instrument's calibrated dial indicates the reciprocal of the "back" focal length, that is, the "back" or *vertex* power. (To add further confusion, "back" vertex power is *also* called the *effective* power of a lens.) In any case, the *vertex* power and not the *true* power is the clinically important one.

Oops! We seem to have drifted off the subject of principle planes for a moment. Back on the track now, I must stress that distances FH and H'F' (the *true* focal lengths) will be equal to each other (as are *both* the focal lengths of any of the thin lenses we have already studied) *if* the media composing both the object and image spaces are of the *same* "refractive index"; (we will get to this term later, in the next section.) On the other hand, if the "refractive index" is *not* the same in both "spaces", the focal length on the side of greater refractive index will be *longer.* More about this later.

NODAL POINTS

We now come to a place where we can treat the nodal points more specifically. By definition, they are a conjugate pair of axial points (the object and image of each other) which have the following property: any ray striking the primary nodal point leaves the secondary nodal point with an identical inclination to the axis. (For the sophisticate, they are points of *unitary angular magnification.*)

If N and N' are nodal points of an optical system, and ray 1 strikes N at $\alpha°$ to the axis, it will leave N' at that same angle.



As long as the medium composing the object space and the medium making up the image space are identical — so far, we have considered only *air* in both "spaces" — the primary *nodal* point is located at precisely the same position as the primary *principal* point, and the secondary nodal point at the secondary principal point, that is, they are superimposed; however, if the media are different (as they are for the eye) both nodal points together shift *away from* superposition with the principal points. How much they shift depends upon the media. They will always shift in the direction of the greater refractive index. One principle is always true: The primary focal length of any optical system is always equal to the distance between N' and the secondary focal point; that is, FH = N'F'. Recall this later when we deal with the eye.

It should now be clear that nodal points (there are no nodal planes) and principal planes and points, as well as focal points, are convenient and important reference positions for *all* optical systems and

all join together to describe completely the focusing action of a particular system. The six axial points we have thus far discussed — (two principal, two nodal, and two focal) are called the *cardinal* points of any optical system. (Two *other* pairs of points which you may see listed in other texts are much less important but are also sometimes included as "cardinal" points — these are the *negative nodal points,* and the *symmetrical points.* Forget them for our study.)

In the complex optical system demonstrated here (as well as in the eye), the primary and secondary principal points are physically separated from one another; (so also would be the primary and secondary nodal points N and N'). The actual separation between H and H' and between N and N' is always *identical* for *any* given optical system; (even when the Ns are not superimposed on the Hs). But, this separation does vary with the nature, complexity and linear separation of the elements making up the complete system. For the human eye, the separation between the primary and secondary principal points (and also the corresponding nodal points) is only 0.3 mm.

So, you might ask, when we began fiddling with our single thin lenses, changing the vergence of object rays (using $U + P = V$) and constructing images by drawing those three particular rays, how come we completely ignored H, H', N and N' as reference points? The answer is, we didn't. With thin lenses, all four of these points coincide with the vertex (axial position) of the lens. It is only with more complex systems that individual attention to these points becomes important. Thus, you should now realize that with thin lenses, even though the four individual cardinal points are not obvious, they are there nonetheless, at all times, huddled together and hidden.

SNELL'S LAW AND REFRACTIVE INDEX

We need to have a few more principles at our fingertips *before* we can indulge ourselves with the "meat" of our course — the optics of the eye, the subject which *really* is what we've been waiting for. This present section, of necessity, is a bit more mathematical, but certainly not offensively so; so, approach it with an open mind.

At the outset, we began by leaping headlong into the concept of vergence and how lenses change that vergence, but we neglected to give you any background in how lenses really work and what governs their activity. As good a place as any to begin is with the basic and fundamental law of optics which makes all lenses and refracting surfaces work to form images — Snell's Law.

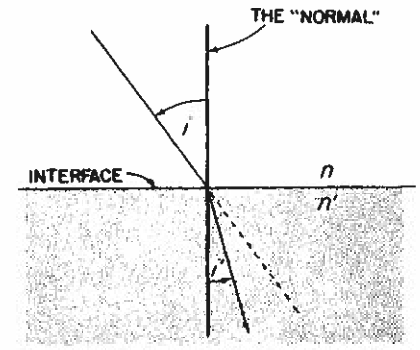
This law concerns itself with exactly to what extent each light ray is bent by surfaces which are separated by media differing in *refractive index*. From your early physics courses, you should remember that this *index* is a property of transparent media; as light passes through any medium other than a vacuum, it is slowed down. The index is simply the ratio of how fast light travels in a vacuum compared to its speed in the specific medium. The refractive index

$$n = \frac{\text{velocity of light in a vacuum}}{\text{velocity of light in the specific medium}}$$

Since the denominator of this fraction is always less than the numerator, this ratio n is always greater than 1 for any medium other than the vacuum (or, more practically, air).

Interestingly, the velocity with which light travels in any medium depends not only on the medium itself but on its own wavelength. Each wavelength has its own "private" index of refraction for each particular medium; the index you usually see listed in tables is for the specific wavelength of sodium light, 589 nm (10^{-9} meters). For water, this particular index is $n = 1.333$, for crown ophthalmic glass (1.523), for plastic (1.491), for the lens of the eye (1.42), and for the cornea (1.376).

When a ray of light which is traveling on one medium hits another medium of a greater index of refraction, that ray of light will be slowed down (and vice versa). If it strikes the material perpendicular ("normal") to its surface, though it is still slowed down, it does not change direction but continues on in the same direction, 90° to the plane of the surface. If, however, the ray of light strikes the new material at some angle (inclination) to the "normal", it will be deviated after crossing the boundary. ***Snell's Law will tell us how much this ray is bent.***



Let n be the index of refraction of the medium on one side of the interface and n' its counterpart — remember that these indices are for a *single* wavelength of light. The angle which an incoming ray makes with the "normal" we will call i. After entering the new medium, the ray is bent — *toward* the normal *if* n' is greater than n. The angle after refraction is denoted by i'. i and i' are called the angles of incidence and refraction, respectively, and are always measured to "the normal", the line drawn perpendicular to the surface at the point where the ray strikes it.

The basic law of Snell — which governs all refraction and forms the basis of how lenses work in their changing the direction of light rays — is as follows:

$$n \sin i = n' \sin i'$$

Simple enough; but, since this represents a trigonometric relationship, let us try to simplify it even further. (I feel it is definitely worthwhile to go through this completely, since you will automatically pick up familiarity with some units you should understand.)

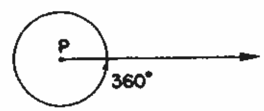
We will begin by looking at how angles are measured.

ANGULAR MEASUREMENT

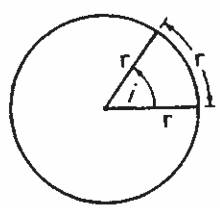
In high school geometry and trigonometry, you were given different units with which to measure angles — the degree and the radian.

When studying the field of strabismus, you will encounter three additional ones — the prism diopter (the only important one), the centrad, and the meter-angle. It would be useful to review each of these units briefly:

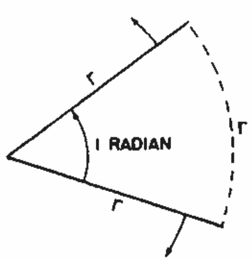
a) *The Degree* — simply defined as 1/360 of the complete angle around a point.

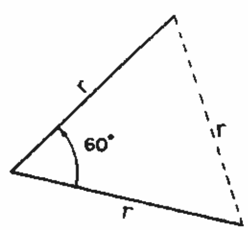


b) *The Radian* — Construct a circle of any radius, and draw two radius arms extending from the center.



The included angle *i* measures *1 radian* when the actual length of the curved arc subtended is equal to the length of the radius.

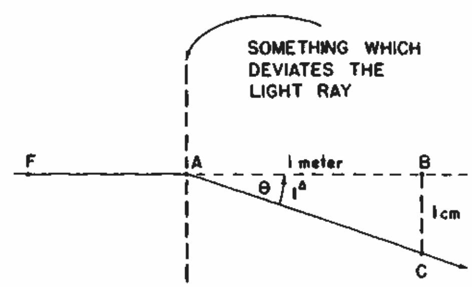




If the circle circumference were made of string and the stiff-radius ends were fixed to it and the included angle was 1 radian, one can see that angle *i* could increase somewhat before the string became taut to a straight line. When this point is reached, *i* would, of course, then be 60°, since the "straightened out arc" would still be equal to the length of the radius. This makes the triangle equilateral (with each angle equal to 60°). So, a measure of 1 radian must be *less* than 60°; let us now see how much less and how we find the exact equivalent.

In any circle there are always a fixed number of radius lengths which (laid end to end) would complete the circumference. That number is 2π; that is, there are (2π) radii which make up the complete circumference of 360° (Remember? $C = 2\pi r$). Since *each r* measured along the circumference represents 1 radian of central *angular* measure (the definition of "radian"), and since there are 2π radii in the complete circumference, there must be 2π radians of angular measure equivalent to 360°. Since 2π radians = 360°, 1 radian must equal 360° divided by 2π (approximately 57.3°); and conversely, 1° would equal $\frac{2\pi}{360}$ radians. The reason we are dragging this unit up from the deep dark past will soon be apparent.

c) *The Prism Diopter* — The angle corresponding to an apparent displacement of 1 cm at 1 meter distance.



Ray FA (after encountering some optical device) is deflected from its "straight ahead" path by angle *θ* towards C. Since triangle ABC is a right triangle, tangent $\theta = \frac{BC}{AB}$.

By definition, when $\theta = 1^{\Delta}$ (prism diopter),

BC = 1 cm and AB = 1 meter.

Then, $\tan \theta = \frac{.01 \text{ meter}}{1 \text{ meter}} = .01$.

By the same reasoning, if θ is 5^{Δ}, $\tan \theta = .05$.

Now you should see that the converse is also true, that is, for any angle θ, the expression *100 tan θ* equals the angle in prism diopters. So, the conversion from degrees to prism diopters (and vice versa) is tied to the *tangent* of the angle, but, herein lies a problem. Look at Table I, which demonstrates the relationship between θ in degrees, the tangent θ (looked up in a standard math table) and the prism diopter:

TABLE 1

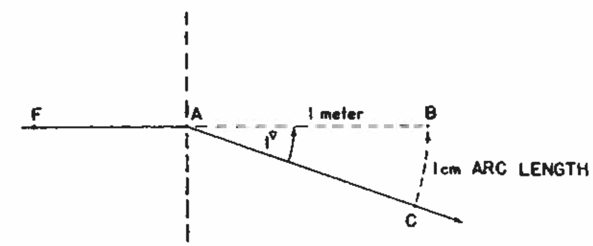
Degrees	*Tan θ*	*100 Tan θ (Prism Diopters)*
1	.01746	1.7
2	.03492	3.5
10	.17633	17.6
20	.36397	36.4
30	.57735	57.7
44	.96569	96.5
45	1.00000	100.0

At small angles, 1° is equivalent to 1.7^{Δ}; at larger angles (say between 44° and 45°) 1° is equivalent to 3.5^{Δ}! What self-respecting kind of *unit* would change its size along a scale? It should be clear, then, that the prism diopter (100 tan θ) is *not* a true unit; it keeps changing its size compared to the degree (which, of course, *is* a true unit).

Don't get me wrong — the prism diopter is a *useful* "unit", but only for measuring *small* angles. In the typical clinical situation when you are measuring, say, esotropia, you usually deal with angles of up to about 30°. In this range of 0° to 30° we can consider 1 degree *approximately* equal to 2 prism diopters; this assumption is actually quite close to being correct. But, keep in mind that the error introduced by using this approximation with larger angles can be large; the "unit"

becomes rapidly meaningless when dealing with angles greater than 45°.

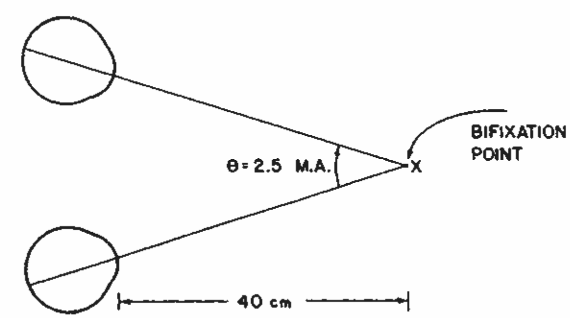
d) *The centrad* ($^{\nabla}$) is similar to the prism diopter except that the 1 cm length displacement (at 1 meter) is measured along the *arc* of a circle instead of a straight segment. For small angles the centrad is approximately equal to the prism diopter.



Now, *forget* the centrad; it is outmoded and obsolete.

e) *The meter-angle:* This angular measure is also not a very important one, but it is encountered in the squint literature in connection with the determination of the ratio between accommodation and its synkinetic accommodative-convergence, that is, the $\frac{AC}{A}$ ratio. (The prism diopter is much more appropriate as the unit for this purpose.)

In any case, when a close-up object is being scrutinized by the eyes, the angle (of convergence) between the two visual axes may be given in meter-angles.



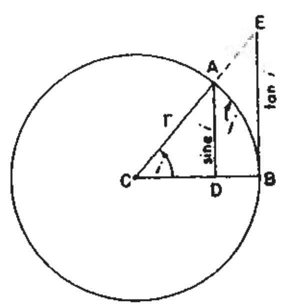
If the eyes converge to point X, angle θ is automatically expressed in meter-angles when the distance of X from the eyes is given in diopters of vergence. So, if X is 40 cm away, angle θ is 2.5 meter-angles.

This "unit" also is not a *true* unit since the angle θ (if it is expressed in degrees or radians) depends on how far apart the eyes are; that is, the true angle *should* depend on the interpupillary distance (p.d.). If the p.d. were greater than that shown here, θ would actually *have* to be greater; yet it still remains only 2.5 meter-angles! The use of "meter-angles" avoids taking account of the actual p.d., and this turns out to be both the advantage and the disadvantage of this "unit".

We would be quite well off without this term too.

The student may question the need for this digression here into the units of angle measurement. In its defense, not only did I want all readers to understand each unit, but I also hoped that this would allow us to begin at the same baseline of information regarding the *radian.*

We have already noted that central angle *i* (in the figure below) can be expressed in degrees or radians. The subtended arc also can be given in either degrees or radians, its measure being exactly equivalent to that of central angle *i* (that is, we can speak of the *arc* itself as equal to, say, 45° or .78 radians) However, the actual measurement (in inches or meters) of the *length* of arc *i* is obviously dependent on the length of the radius. With the same central angle, the longer the radius, the longer the length of arc *i*. If the angle *i* is 1 radian, arc *i* exactly equals the length of the radius; if angle $i = 0.5$ radians, arc *i* equals one-half the length of the radius. Thus, arc *i* always equals the central angle *i* (in radians) *times* the length of the radius.



We have drawn the circle above with its center at C and two radii, CA and CB, and constructed two perpendicular lines to radius CB; one is dropped from the end of radius CA, the other at the end of radius CB.

$$\sin i = \frac{AD}{AC} = \frac{AD}{r} \qquad \tan i = \frac{EB}{CB} = \frac{EB}{r}$$

If we assume r = 1 "unit" of any length,

then $\sin i = AD$

and $\tan i = EB.$

In other words, lines AD and EB are the linear representations of the sine and tangent of angle *i*, respectively.

As we have shown, arc AB in the above figure equals *i* (in radians) times the length of the radius. Since the radius is equal to 1, arclength AB is equivalent to angle *i*.

From the diagram, let us now list these three lengths in order of size: AD is shortest, then arclength AB, and longest is BE; thus, by substitution

$$\sin i < i < \tan i.$$

You should now be able to picture what happens as angle *i* decreases; these three become just about equal to each other. So, for relatively small angles, *i* (in radians) is a very acceptable substitute for the trigonometric expressions sin *i* and tan *i*. To show you how little error is introduced by such a substitution, I have constructed Table II.

TABLE II

i in degrees	sin *i* (from math tables)	*i* in radians	tan *i* (from math tables)	% error introduced by using *i* (radians) instead of sin *i*	% error introduced by using *i* (radians) instead of tan *i*
1°	.01745	.01745	.01746	0.00%	0.01%
2°	.03490	.03490	.03492	0.00%	0.01%
10°	.17365	.17450	.17633	+ 0.49%	− 1.04%
20°	.34202	.34900	.36397	+ 2.01%	− 4.12%
30°	.50000	.52350	.57735	+ 4.5 %	− 9.35%
45°	.70711	.78525	1.00000	+11.1 %	−21.4 %

This table shows the percentage error introduced by substituting i (in radians) for sin i and for tan i for various angular measures. Three points should be clear:

1) i is an excellent substitute for sin i and tan i for small angles, probably up to 20°. Even for angles as large as 20°, i errs by only 2% when substituted for sin i and 4% for tan i.

2) i is a somewhat better substitute for sin i than for tan i.

3) When one substitutes i for sin i, the approximation will be slightly too large; when using i for tan i, it will be slightly too small. (This is also evident by looking at the last figure, specifically at the lengths AD, $\widehat{AB}$, and BE; if you compare these lines you will have a *graphical* demonstration of our approximation i for sin i and tan i.)

We can investigate this approximation in still another way. When any angle i is expressed in radians, we can find the value of sin i by simply substituting this value of i in a mathematical "trigonometric expansion" which is the mathematical equivalent of sin i. (Incidentally, this is how the math tables themselves are constructed.) Don't close your eyes! It's not that scary. That expansion is as follows:

$$\sin i = i - \frac{i^3}{3!} + \frac{i^5}{5!} - \frac{i^7}{7!} + \ldots \quad (3! = 3 \times 2 \times 1)$$
$$(5! = 5 \times 4 \times 3 \times 2 \times 1)$$

Also,

$$\tan i = i + \frac{i^3}{3} + \frac{2\,i^5}{15} + \frac{17\,i^7}{315} + \ldots$$

We can thus find the value of sin i or tan i as accurately and to as many decimal places as we wish. The question is how many of those $\frac{i^n}{n!}$ terms (for sin i) must be used to be reasonably accurate. The obvious answer, as we have shown in Table II, is that if we limit ourselves to i of *small* angles, we can leave off all the terms beyond the first one; we make almost no error in stating that sin i is equal to angle i itself (when i is expressed in radians). *This* is the approximation we make in what is called *first order optics*. It is somewhat analogous to using the approximation 3.1 for π; if you need a more accurate value, you could add another term, as 3.14. Likewise, if you need a more accurate value for sin i, you can include the second term $(-\frac{i^3}{3!})$ too.

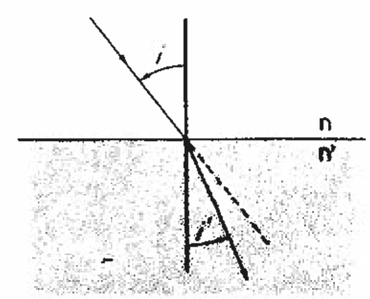
When we do use the more refined value $(i - \frac{i^3}{3!})$ for sin i in optical calculations, we are using *"third order" optics* — note there is no "2nd order". Comparable terms can also be added for tan i to increase its accuracy. The additional degree of accuracy for both sin i and tan i is necessary to account for some of the lens aberrations. Astronomers may require 5th or even 7th order optics for their accurate calculations of star positions. But, the point of importance for us clinicians is that for everything we will learn here, we really need only the 1st term (i) to substitute for sin i.

First order optics includes the evaluation of object and image rays — called *paraxial* rays — which lie close to the axis of refracting systems, so that angles of incidence and refraction are relatively small. In spite of the fact that we use large angular diagrams in this book to elaborate certain principles, we must realize we are still only describing *accurately* the action of lenses on the *paraxial* rays.

Later, when we deal with curved refracting *surfaces*, we will substitute i for sin i; Snell's Law will become simplified to $n\, i = n' i'$, and correctly so, as long as we do not deal with too large angles and we express i in radians.

PLANE SURFACE REFRACTION

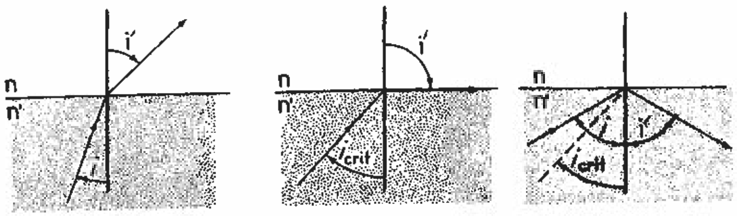
Snell's Law governs the refraction of light rays. *Without* using any approximations now, we can say, $n \sin i = n' \sin i'$.



A light ray passing from a less dense medium to a denser one (with a greater index n') is bent *toward* the "normal"; and conversely, light moving the other direction (from n' toward n) is bent *away* from the normal. (Remember that all light rays are retraceable.)

"Critical Angle"

When light emanates from an object point located within the denser medium (figure A below), the stage is set for a peculiar phenomenon to occur.



As angle i increases, so does angle i'. Angle i will eventually reach a certain magnitude such that i', the angle of *refraction*, will become equal to 90° (figure B above). At this position, i (the angle of incidence) is called the *"critical"* angle. If i now is increased further, even a slight amount, the ray will *not* exit from medium n' at all; it will be totally reflected* internally (as shown in C above).

Since, in this example,

$$n' \sin i = n \sin i'$$

$$\sin i = \frac{n}{n'} (\sin i')$$

By definition, at i_{crit} (the "critical" angle)

$$i' = 90°, \text{ thus } \sin i' = 1.00.$$

Therefore, $$\sin i_{crit} = \frac{n}{n'}$$

When n' represents water ($n' = 1.33$) and $n = 1.00$ for air,

$$\sin i_{crit} = \frac{1}{1.33} = 0.75$$

$$i_{crit} = 48°.$$

* We will take up "reflection" later.

Hence, the critical angle for water is 48°, but only for a *water-to-air* interface. Similarly, the sine of the critical angle for corneal tissue $= \frac{1}{1.376}$;

$$\sin i = 0.725$$

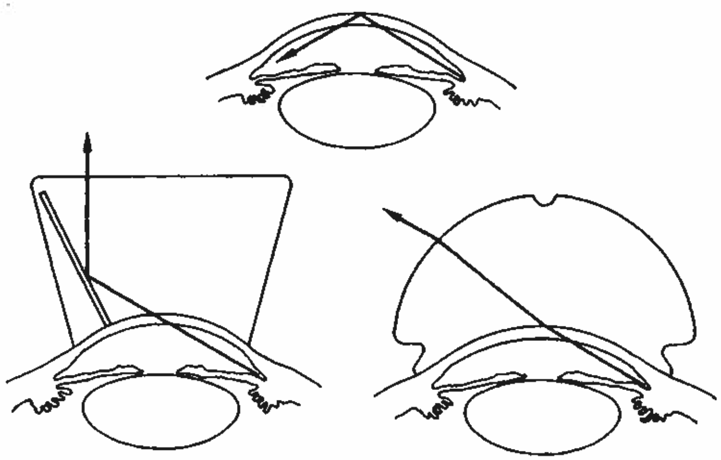
$$i_{crit} = 46.5°$$

This point has clinical importance. Recall that, if we are to see any object at all, light rays from it must enter our eyes. Consider the situation of the anterior chamber angle; light rays which come from the angle must pass out through the cornea. They do pass and are refracted by the posterior corneal surface; however, because of the particular dimensions of the anterior chamber angle and its distance from the cornea, the light rays which leave there strike the *anterior* corneal surface (an interface with a greater index of refraction *inside* than outside) at an angle of incidence *greater* than the critical angle of corneal tissue (∠46°). So, all these rays are reflected back into the eye. Since they are unable to escape, a clinician normally is unable to visualize the anterior chamber angle of a patient. If the cornea happens to be "steeper" than normal (as in a patient with keratoconus), the rays may strike the interface with *less* angular incidence than the critical angle and thus might be able to leave the eye. So occasionally, the chamber angle may be seen by an observer, but not usually. (See A next figure.)

Diagnostic Goniolenses

In the *typical* patient, the angle can only be visualized with optical help — by "optically" removing the corneal front surface and replacing it with a new surface (or one with a *different* curvature) which allows the light to escape. This can be done with a contact lens whose own index of refraction is substituted for that of the air. This decreases the *difference* in index of refraction across that interface and thus, optically "removes" the original corneal surface. It works by the same principal which causes a glass marble to disappear when it is immersed in a dish of water. The glass ball is easily visible in air because one "sees" the surface due to the marked difference in index between the glass and the air. But, when it is put into water, the indices are so close that the marble's surface is optically eliminated. *Therapeutic* contact lenses* which correct corneal irregularity (as in keratoconus) or even correct simple refractive error also work in the same way, with any necessary prescription "corrective power" being ground onto the front surface of the lens.

A. TOTAL INTERNAL REFLECTION



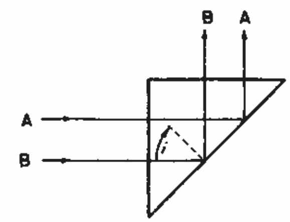
B. GOLDMANN LENS   C. KOEPPE LENS

Two main types of diagnostic contact lenses are used for gonioscopy, the viewing of the chamber angle: a Koeppe contact lens allows *direct* viewing of the angle (see Figure C; and a Goldmann type lens incorporates a mirror so that an examiner sees a *reflected* or *indirect* view of the anterior chamber angle on the opposite side (that is, an upper mirror allows view of the lower angle). (See figure B above.)

The presence of a critical angle is not always a disadvantage; it *can* be used constructively. For their useful operation, many optical instruments require that a light beam change its direction. An ophthalmoscope is a good example. It requires "something" to bend the light

* The subject of therapeutic corneal contact lenses is so broad that it would be out of place in this book and will not be discussed further here. Excellent texts (such as *Corneal Contact Lenses* by Girard, Soper and Sampson, Mosby, 1970) are available for consultation and reference in this important clinical area.

beam which runs parallel to the handle and change its direction by 90°. This allows you to shine the beam into a patient's eye while you are observing the fundus. Another example is the typical binocular telescope ("binoculars"); this instrument would be much longer in dimension were it not for the fact that the light path is "folded" inside. The changes of light direction in both these instruments is accomplished by prisms which totally *reflect* light (as would a mirror, but more efficiently — with *less* light loss).



This total reflection is shown in the accompanying diagram above. Since the angle of incidence *i* exceeds the "critical angle" of this prism material, the light rays are totally reflected internally. For special purposes, many other prism forms are available which not only totally reflect light but also twist and reorient the image as well. (e.g. Porro and Abbe prisms.)

There are many other useful applications of total internal reflection, for example, "fiber optic" bundles.